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On the Symmetric Form of Systems of Conservation Laws with Entropy*

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This paper reviews the symmetrizability of systems of conservation laws which possess entropy functions. Symmetric formulations in conservation form for the equations of gas dynamics are presented.

INTRODUCTION

In this paper we consider systems of conservation laws which possess an entropy function. Such equations of mathematical physics can be written in a symmetric form which retains the conservation properties of the system. Among the researchers who have investigated this class of equations are Godunov [2], Friedrichs and Lax [3], and more recently Mock [6] and Harten and Lax [4].

The symmetrizability of systems of conservation laws with entropy may and should be utilized in the design and analysis of numerical solutions to such problems. For example, it offers the possibility of locally linearizing the equations in a way which preserves the hyperbolicity and conservation properties (see Roe [7, 8], the next section, and [5]). Another example is the use of the symmetrizibility property to rigorously analyze splitting algorithms for the Navier–Stokes equations by Abarbanel and Gottlieb (see [1]). Of particular interest is the possibility of improving the structure of iteration matrices in direct Newton-iteration methods to the solution of the steady state equations.

The goal of this paper is to review the general structure of systems of conservation laws with entropy, and in particular to present symmetric formulations of the equations of gas dynamics. It is hoped that this information will be of service to the designers of numerical approximations of this important class of equations.

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1. Systems of Conservation Laws with Entropy

In this paper we consider systems of hyperbolic conservation laws of the form

$$u_t + \sum_{i=1}^{d} f^i(u)_{x_i} \equiv u_t + \operatorname{div} f(u) = 0. \tag{1.1a}$$

Here $u(x, t)$ is an m-column vector of unknowns, $f^i(u)$ is a vector-valued function of m components, $x = (x_1, ..., x_d)$, and $f = (f^1, ..., f^d)$. We can write (1.1a) in the matrix form

$$u_t + \sum_{i=1}^{d} A^i(u)\, u_{x_i} = 0, \tag{1.1b}$$

where

$$A^i(u) = f^i_u. \tag{1.1c}$$

Equation (1.1) is called hyperbolic if the matrix

$$\sum_{i=1}^{d} \omega_i A^i(u) \tag{1.2}$$

has real eigenvalues and a complete set of eigenvectors for all real ω_i.

A scalar function $U(u)$ is an entropy function for (1.1) if

(i) The function U satisfies

$$U_u f^i_u = F^i_u, \qquad i = 1, ..., d, \tag{1.3}$$

where $F^i(u)$ is some scalar function called entropy flux in the x_i direction.

(ii) The function U is a convex function of u.

It follows from (1.3), upon multiplication of (1.1a) by U_u, that every smooth solution of (1.1) also satisfies

$$U_t + \sum_{i=1}^{d} F^i_{x_i} \equiv U_t + \operatorname{div} F = 0, \tag{1.4}$$

where $F = (F^1, ..., F^d)$.

A system of equations

$$P v_t + \sum_{i=1}^{d} B^i v_{x_i} = 0, \tag{1.5}$$

is called symmetric hyperbolic if P and all B^i are symmetric matrices, and if P is positive definite.

The symmetrization of (1.1) will be accomplished by introducing new dependent variables v in place of u by setting $u = u(v)$, i.e.,

$$u(v)_t + \sum_{i=1}^{d} f^i(u(v))_{x_i} = u_v v_t + \sum_{i=1}^{d} f^i_v v_{x_i} = 0. \tag{1.6a}$$

Thus (1.1) becomes of form (1.5) with

$$P = u_v, \qquad B^i = f^i_v. \tag{1.6b}$$

The symmetry of the matrices u_v and f^i_v implies that u and f^i are gradients with respect to v, i.e., there exist scalar functions $q(v)$, $r^i(v)$ such that

$$q_v = u^{\mathrm{T}}, \tag{1.7a}$$

$$r^i_v = (f^i)^{\mathrm{T}}, \tag{1.7b}$$

where superscript T denotes transpose. The positive definiteness of u_v is equivalent to the convexity of $q(v)$.

Note that the convexity of q implies that the mapping $v \to q_v$ is one-to-one, so that (1.7a) can be inverted, i.e., v can be regarded as a function of u.

THEOREM 1.1 (Godunov). *Suppose* (1.1) *can be symmetrized by introducing new variables v, i.e.,* (1.7) *holds, where q is a convex function of v. Then* (1.1) *has an entropy function $U(u)$ given by*

$$U(u) = u^{\mathrm{T}} v - q(v), \tag{1.8a}$$

with entropy fluxes $F^i(u)$

$$F^i(u) = (f^i)^{\mathrm{T}}\, v - r^i(v). \tag{1.8b}$$

Proof. Differentiate (1.8a) with respect to u; using (1.7a), we get

$$U_u = v^{\mathrm{T}} + u^{\mathrm{T}} v_u - q_v v_u = v^{\mathrm{T}}. \tag{1.9}$$

Similarly, from (1.8b) and (1.7b) we get

$$F^i_u = v^{\mathrm{T}} f^i_u + (f^i)^{\mathrm{T}}\, v_u - r^i_v v_u = v^{\mathrm{T}} f^i_u. \tag{1.10}$$

Relation (1.3) follows.

To prove the convexity of U, we show that U is the Legendre transform of q:

$$U(u) = \max_v\, [u^{\mathrm{T}} v - q(v)]. \tag{1.11}$$

For, by the convexity of q, the right side has a unique maximum; at the maximum point, the v derivative must vanish; this gives relation (1.7a). This proves that (1.11) is the same as (1.8a). Equation (1.11) represents U as the maximum of linear functions; this proves that U is convex.

Conversely:

THEOREM 1.2 (Mock). *Suppose $U(u)$ is an entropy function for* (1.1); *then*

$$v^{\mathrm{T}} = U_u, \tag{1.12}$$

symmetrizes (1.1).

Proof. The convexity of U implies that the mapping $u \to U_u$ is one-to-one, hence (1.12) defines u as a function of v. We define now q and r^i by

$$q(v) = v^{\mathrm{T}} u - U(u), \tag{1.13a}$$

$$r^i(v) = v^{\mathrm{T}} f^i - F^i(u), \tag{1.13b}$$

where F^i are the entropy fluxes. Differentiating (1.13a) with respect to v, and using (1.12) gives

$$qv = u^T + v^T u_v - U_u u_v = u^T.$$

Similarly, from (1.13b), (1.12), and (1.3) we get

$$r^i_v = (f^i)^{\mathrm{T}} + v^{\mathrm{T}} f^i_u u_v - F^i_u u_v = (f^i)^{\mathrm{T}}.$$

These formulas show that (1.7a) and (1.7b) hold; therefore u_v and f_v are symmetric. To show that u_v is positive, we have to verify that q is convex. This can be done, as before, by observing that, because of the convexity of U, it follows from (1.13a) and (1.12) that q is the Legendre transform of U. (For more details see [4].)

We note the following relations:

(i) The symmetric positive definite matrix u_v simultaneously symmetrizes all $A^i = f^i_u$ from the right, i.e.,

$$A^i u_v = B^i = \text{symmetric}. \tag{1.14a}$$

(ii) The symmetric positive definite matrix v_u simultaneously symmetrizes all A^i from the left.

$$v_u A^i = v_u B^i v_u = \text{symmetric}. \tag{1.14b}$$

(iii) The similarity transformation

$$(v_u)^{1/2} A^i (v_u)^{-1/2} = (v_u)^{1/2} B^i (v_u)^{1/2} = \text{symmetric} \tag{1.14c}$$

simultaneously transforms all A^i into symmetric matrices.

We say that the system (1.1) can be linearized in the sense of Roe if for all u_1 and u_2 there exist matrices $A^i(u_1, u_2)$ such that for $i = 1,..., d$

(i) $f^i(u_2) - f^i(u_1) = A^i(u_1, u_2)(u_2 - u_1)$, (1.15a)

(ii) $A^i(u, u) \equiv f^i_u(u) \equiv A^i(u)$, (1.15b) and

(iii) the matrix

$$\sum_{i=1}^{d} \omega_i A^i(u_1, u_2) \tag{1.15c}$$

has real eigenvalues and a complete set of eigenvectors for all real ω_i (see [7, 8]).

THEOREM 1.3 (Harten–Lax). *Suppose* (1.1) *has an entropy function*; *then* (1.1) *can be linearized in the sense of Roe.*

Proof. Let $v^{\mathrm{T}} = U_u$. Then by Theorem 1.2, the mapping $u \to v$ is one-to-one, v_u is a symmetric positive definitive matrix, and the f^i_v are symmetric. Let $v_1 = v(u_1)$, $v_2 = v(u_2)$ and define

$$v(\theta) = v_1 + \theta(v_2 - v_1).$$

Then

$$f^i(u_2) - f^i(u_1) = \int_0^1 f^i_v(v(\theta)) \frac{dv}{d\theta} d\theta = \int_0^1 f^i_v(v(\theta))\, d\theta (v_2 - v_1).$$

Denote

$$B^i(u_1, u_2) = \int_0^1 f^i_v(v(\theta))\, d\theta; \tag{1.16a}$$

then

$$f^i(u_2) - f^i(u_1) = B^i(u_1, u_2)(v_2 - v_1), \tag{1.16b}$$

where $B^i(u_1, u_2)$ is symmetric. Now let

$$u(\eta) = u_1 + \eta(u_2 - u_1).$$

Then

$$v_2 - v_1 = \int_0^1 v_u(u(\eta)) \frac{du}{d\eta} d\eta = \int_0^1 v_u(u(\eta))\, d\eta (u_2 - u_1). \tag{1.17a}$$

Denote

$$P(u_1, u_2) = \int_0^1 v_u(u(\eta))\, d\eta;$$

then

$$v_2 - v_1 = P(u_1, u_2)(u_2 - u_1), \tag{1.17b}$$

where $P(u_1, u_2)$ is symmetric positive definite. Combining (1.16b) and (1.17b), we get

$$f^i(u_2) - f^i(u_1) = A^i(u_1, u_2)(u_2 - u_1), \tag{1.18}$$

where

$$A^i(u_1, u_2) = B^i(u_1, u_2)\, P(u_1, u_2). \tag{1.19}$$

For $u_1 = u_2 = u$, we get that $v(\theta) \equiv v(u)$, $u(\eta) \equiv u$, and $B(u_1, u_2) = f_v^i(v(u))$, $P(u_1, u_2) = v_u(u)$. Hence

$$A^i(u, u) = B^i(u, u)\, P^i(u, u) = f_v^i(v(u))\, v_u(u) = f_u^i(u) = A^i(u). \tag{1.20a}$$

Denote

$$C = \sum_{i=1}^{d} \omega_i A^i(u_1, u_2) \equiv \left[\sum_{i=1}^{d} \omega_i B^i(u_1, u_2)\right] P(u_1, u_2).$$

Then

$$\begin{aligned}[P(u_1, u_2)]^{1/2}\, &C[P(u_1, u_2)]^{-1/2} \\ &= [P(u_1, u_2)]^{1/2} \left[\sum_{i=1}^{d} \omega_i B^i(u_1, u_2)\right] [P(u_1, u_2)]^{1/2} = \text{symmetric}.\end{aligned} \tag{1.20b}$$

Thus C is similar to a symmetric matrix and therefore has real eigenvalues and a complete set of eigenvectors for all real ω_i.

2. Euler Equations of Gas Dynamics

In this section we consider the Euler equations for polytropic gas in conservation form:

$$u_t + [f^x(u)]_x + [f^y(u)]_y = 0, \tag{2.1a}$$

where

$$u^{\mathrm{T}} = (\rho, m, n, E) \equiv (u_1, u_2, u_3, u_4),$$

$$\begin{aligned}[f^x(u)]^T = u_1^{-2}\{&u_1^2 u_2, (\gamma - 1)\, u_1(u_1 u_4 - \tfrac{1}{2}u_3^2) + \tfrac{1}{2}(3 - \gamma)\, u_1 u_2^2, u_1 u_2 u_3, \\ &u_2[\gamma u_1 u_4 - \tfrac{1}{2}(\gamma - 1)(u_2^2 + u_3^2)]\},\end{aligned} \tag{2.1b}$$

$$\begin{aligned}[f^y(u)]^T = u_1^{-2}\{&u_1^2 u_3, u_1 u_2 u_3, (\gamma - 1)\, u_1(u_1 u_4 - \tfrac{1}{2}u_2^2) + \tfrac{1}{2}(3 - \gamma)\, u_1 u_3^2, \\ &u_3[\gamma u_1 u_4 - \tfrac{1}{2}(\gamma - 1)(u_2^2 + u_3^2)]\},\end{aligned} \tag{2.1c}$$

where ρ is the density, E is the total energy, m and n are the momentum in the x direction and y direction, respectively, and $\gamma > 1$ is the adiabatic exponent.

The Jacobian matrix $A^x = f^x_u$ equals

$$-u_1^{-3}\begin{bmatrix} 0 & -u_1^3 & 0 & 0 \\ [\frac{1}{2}(3-\gamma)\,u_2^2 + \frac{1}{2}(1-\gamma)\,u_3^2]\,u_1 & (\gamma-3)\,u_1^2 u_2 & (\gamma-1)\,u_1^2 u_3 & (1-\gamma)\,u_1^3 \\ u_1 u_2 u_3 & -u_1^2 u_3 & -u_1^2 u_2 & 0 \\ [\gamma u_1 u_4 + (1-\gamma)\cdot(u_2^2+u_3^2)]u_2 & [-\gamma u_1 u_4 + \frac{1}{2}(\gamma-1)\cdot(3u_2^2+u_3^2)]u_1 & (\gamma-1)\,u_1 u_2 u_3 & -\gamma u_1^2 u_2 \end{bmatrix} \tag{2.2a}$$

and has eigenvalues

$$a_1^x = u_1^{-1}\{u_2 - [\gamma(\gamma-1)]^{1/2}\,[u_1 u_4 - \tfrac{1}{2}(u_2^2+u_3^2)]^{1/2}\}; \qquad a_2^x = a_3^x = u_1^{-1} u_2;$$
$$a_4^x = u_1^{-1}\{u_2 + [\gamma(\gamma-1)]^{1/2}\,[u_1 u_4 - \tfrac{1}{2}(u_2^2+u_3^2)]^{1/2}\}. \tag{2.2b}$$

The Jacobian matrix $A^y = f^y_u$ equals

$$-u_1^{-3}\begin{bmatrix} 0 & 0 & -u_1^3 & 0 \\ u_1 u_2 u_3 & -u_1^2 u_3 & -u_1^2 u_2 & 0 \\ u_1(3-\gamma)\,u_3^2 + \frac{1}{2}(1-\gamma)\,u_2^2) & (\gamma-1)\,u_1^2 u_2 & (\gamma-3)\,u_1^2 u_3 & (1-\gamma)\,u_1^3 \\ [\gamma u_1 u_4 + (1-\gamma)\cdot(u_2^2+u_3^2)]u_3 & (\gamma-1)\,u_1 u_2 u_3 & [-\gamma u_1 u_4 + \frac{1}{2}(\gamma-1)\cdot(3u_3^2+u_2^2)]u_1 & -\gamma u_1^2 u_3 \end{bmatrix} \tag{2.3a}$$

and has eigenvalues

$$a_1^y = u_1^{-1}\{u_3 - [\gamma(\gamma-1)]^{1/2}\,[u_1 u_4 - \tfrac{1}{2}(u_2^2+u_3^2)]^{1/2}\}; \qquad a_2^y = a_3^y = u_1^{-1} u_3;$$
$$a_4^y = u_1^{-1}\{u_3 + [\gamma(\gamma-1)]^{1/2}\,[u_1 u_4 - \tfrac{1}{2}(u_2^2+u_3^2)]^{1/2}\}. \tag{2.3b}$$

It is well known that (2.1) implies that

$$S = \log[P\rho^\gamma] = \log\{(u_1^{-\gamma-1}/(\gamma-1))[u_1 u_4 - \tfrac{1}{2}(u_2^2+u_3^2)]\} \tag{2.4a}$$

(where

$$P = (\gamma-1)u_1^{-1}[u_4 u_1 - \tfrac{1}{2}(u_2^2+u_3^2)] \tag{2.4b}$$

is the pressure) satisfies

$$u_1(dS/dt) = u_1 S_t + u_2 S_x + u_3 S_y = 0$$

for all smooth $u(x, t)$, i.e., S is constant along streamlines.

Consequently

$$u_1 h(S)_t + u_2 h(S)_x + u_3 h(S)_y = u_1 \dot{h}(S)\frac{dS}{dt} = 0 \tag{2.5a}$$

for all differentiable functions $h(S)$. Here $\dot{h}$ denotes derivative with respect to S.

Multiplying the continuity equation in (2.1)

$$u_{1t} + u_{2x} + u_{3y} = 0 \tag{2.5b}$$

by $-h(S)$ and subtracting (2.5a), we obtain the entropy equation (1.4) for (2.1).

$$[-u_1 h(S)]_t + [-u_2 h(S)]_x + [-u_3 h(S)]_y = 0. \tag{2.6a}$$

Here

$$U(u) = -u_1 h(S), \qquad F^x(u) = -u_2 h(S), \qquad F^y(u) = -u_3 h(S). \tag{2.6b}$$

Then $v^{\mathrm{T}} \equiv (v_1, v_2, v_3, v_4)$ in (1.12) becomes

$$v^{\mathrm{T}} = -(\gamma-1)[\dot{h}(S)/P]\{u_4 + [P/(\gamma-1)][h(S)/\dot{h}(S) - \gamma - 1], -u_2, -u_3, u_1\}. \tag{2.7}$$

and

$$v_u = [(\gamma-1)/P]^2 u_1 \dot{h}(S)\cdot D \tag{2.8a}$$

where

$$D = \begin{bmatrix} \tfrac{1}{4}q^4 + c_*^4/\gamma - R(\tfrac{1}{2}q^2 - c_*^2)^2 & -q_1[\tfrac{1}{2}q^2(1-R) + Rc_*^2] & -q_2[\tfrac{1}{2}q^2(1-R) + Rc_*^2] & \tfrac{1}{2}q^2(1-R) - c_*^2(1/\gamma - R) \\ -q_1[\tfrac{1}{2}q^2(1-R) + Rc_*^2] & q_1^2(1-R) + c_*^2/\gamma & q_1 q_2(1-R) & -q_1(1-R) \\ -q_2[\tfrac{1}{2}q^2(1-R) + RC_*^2] & q_1 q_2(1-R) & q_2^2(1-R) + c_*^2/\gamma & -q_2(1-R) \\ \tfrac{1}{2}q^2(1-R) - c_*^2(1/\gamma - R) & -q_1(1-R) & -q_2(1-R) & 1-R \end{bmatrix} \tag{2.8b}$$

Here P is the pressure from (2.4b), $c_*^2 = [\gamma/(\gamma-1)]\,P/u_1$, $q_1 = u_2/u_1$, $q_2 = u_3/u_2$, $q^2 = q_1^2 + q_2^2$, and $R = \ddot{h}(S)/\dot{h}(S)$.

We show now that the symmetric matrix D is positive definite if and only if

$$R = \ddot{h}(S)/\dot{h}(S) < 1/\gamma. \tag{2.8c}$$

We do so by showing that the determinants of the major blocks of D are positive if and only if (2.8c) holds.

$$M_{11} = D_{11} = (1/\gamma - R)(\tfrac{1}{2}q^2 - c_*^2)^2 + q^2[\tfrac{1}{4}(\gamma - 1)\, q^2 + c_*^2]/\gamma > 0, \tag{2.9a}$$

$$M_{22} = \det \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{2.9b}$$

$$= (c_*^2/\gamma^2)\{(1 - R\gamma)[(\tfrac{1}{2}q^2 - c_*^2)^2 + (\gamma + 1)\, c_*^2 q_1^2] + c_*^2 q_2^2 + \tfrac{1}{4}(\gamma - 1)\, q^4\} > 0,$$

$$M_{33} = \det \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}$$

$$= (c_*^4/\gamma^2)[(1 - R\gamma)\, c_*^2(q^2 + c_*^2/\gamma) + (1 - R)\, q^4/4] > 0, \tag{2.9c}$$

$$M_{44} = \det(D) = (c_*^8/\gamma^4)(\gamma - 1)(1 - R\gamma) > 0. \tag{2.9d}$$

We note that for the physical entropy (2.4), $h(S) = S$, $\dot{h}(S) \equiv 1$ and $R = 0$. It follows from (2.8c) that v_u is positive definite in this case.

Next we derive specific formulae for the family of functions $h(S)$ defined with respect to a parameter α:

$$h(S) = Ke^{S/(\alpha + \gamma)} = K(P\rho^{-\gamma})^{1/(\alpha + \gamma)}. \tag{2.10a}$$

In this case $\dot{h}(S) = [K/(\alpha + \gamma)]\, e^{S/(\alpha + \gamma)}$; $R = \ddot{h}(S)/\dot{h}(S) = 1/(\alpha + \gamma)$. It follows from (2.8) that v_u is positive definite if and only if

$$\alpha > 0, \qquad K > 0. \tag{2.10b}$$

We note that $\det(v_u) = 0$ if and only if $\alpha = 0$.

Substituting (2.10a) with $K = ((\alpha + \gamma)/(\gamma - 1))$ for $h(S)$ in (2.7), we get

$$v^{\mathrm{T}} = -P^{1/(\alpha + \gamma) - 1} u_1^{-\gamma/(\alpha + \gamma)}\{u_4 + [P/(\gamma - 1)](\alpha - 1), -u_2, -u_3, u_1\}. \tag{2.11}$$

Denote:

$$w \equiv -v, \tag{2.12a}$$

$$\mu = [(\gamma - 1)/\alpha][w_1 w_4 - \tfrac{1}{2}(w_2^2 + w_3^2)]; \tag{2.12b}$$

then $u(v) \equiv u(-w)$ is given by

$$u_1 = \rho = w_4^{(\gamma + \alpha - 2)/(\gamma - 1)} \mu^{(1 - \alpha - \gamma)/(\gamma - 1)}, \tag{2.13a}$$

$$u_2/u_1 = q_1 = -w_2/w_4, \tag{2.13b}$$

$$u_3/u_1 = q_2 = -w_3/w_4, \tag{2.13c}$$

$$(\gamma - 1)[u_1 u_4 - \tfrac{1}{2}(u_2^2 + u_3^2)]/u_1 = p = w_4^{-2} \mu\rho. \tag{2.13d}$$

We turn now to express the fluxes f^x and f^y in (2.1) in terms of the dependent variable v.

$$[f^x(v)]^{\mathrm{T}} = \rho w_4^{-3}\{-w_2 w_4^2,\ w_4(w_2^2+\mu),\ w_2 w_3 w_4,\ -w_2\,[w_1 w_4 - \mu(\alpha-\gamma)/(\gamma-1)]\}, \tag{2.14a}$$

$$[f^y(v)]^{\mathrm{T}} = \rho w_4^{-3}\{-w_3 w_4^2,\ w_2 w_3 w_4,\ w_4(w_3^2+\mu),\ -w_3\,[w_1 w_4 - \mu(\alpha-\gamma)/(\gamma-1)]\}; \tag{2.14b}$$

$\rho(v)$ is given in (2.13a). We observe that the fluxes $f^x(v)$ and $f^y(v)$ are homogeneous functions of v of degree s, i.e.

$$f^x(\beta v) = \beta^s f^x(v), \qquad f^y(\beta v) = \beta^s f^y(v) \tag{2.15a}$$

for all scalar β, where

$$s = -(\alpha+\gamma)/(\gamma-1). \tag{2.15b}$$

We denote

$$k_1 = (1-\alpha-\gamma)/\alpha; \qquad k_2 = (\alpha-\gamma)/(\gamma-1). \tag{2.16}$$

The Jacobian $f_v^x = -f_w^x$ takes the form $f_v^x = -\rho\mu^{-1} w_4^{-2}\cdot \hat{B}^x$ where

$$\hat{B}^x = \begin{bmatrix} -k_1 w_2 w_4^2 & w_4(k_1 w_2^2-\mu) & k_1 w_2 w_3 w_4 & \begin{matrix}-w_2[(k_2+1)\mu \\ +k_1 w_1 w_4]\end{matrix} \\ w_4(k_1 w_2^2-\mu & w_2(3\mu - k_1 w_2^2) & w_3(\mu - k_1 w_2^2) & \begin{matrix}k_2 w_4^{-1}\mu(w_2^2+\mu) \\ +w_1(k_1 w_2^2-\mu)\end{matrix} \\ k_1 w_2 w_3 w_4 & w_3(\mu - k_1 w_2^2) & w_2(\mu - k_1 w_3^2) & \begin{matrix}w_2 w_3(k_2 w_4^{-1}\mu \\ +k_1 w_1)\end{matrix} \\ \begin{matrix}-w_2[(k_2+1)\mu \\ +k_1 w_1 w_4]\end{matrix} & \begin{matrix}k_2 w_4^{-1}\mu(w_2^2+\mu) \\ +w_1(k_1 w_2^2-\mu)\end{matrix} & \begin{matrix}w_2 w_3(k_2 w_4^{-1}\mu \\ +k_1 w_1)\end{matrix} & \begin{matrix}-w_2[w_1(2k_2 w_4^{-1}\mu \\ +k_1 w_1) - k_2(k_2-1)w_4^{-2}\mu^2]\end{matrix} \end{bmatrix}. \tag{2.17a}$$

Similarly the Jacobian $f_v^y \equiv -f_w^y$ takes the form $f_v^y = -\rho^{\mu-1} w_4^{-2}\cdot \hat{B}^y$ where

$$\hat{B}^y = \begin{bmatrix} -k_1 w_3 w_4^2 & k_1 w_2 w_3 w_4 & w_4(k_1 w_3^2-\mu) & \begin{matrix}-w_3[(k_2+1)\mu \\ +k_1 w_1 w_4]\end{matrix} \\ k_1 w_2 w_3 w_4 & w_3(\mu - k_1 w_2^2) & w_2(\mu - k_1 w_3^2) & \begin{matrix}w_2 w_3(k_2 w_4^{-1}\mu \\ +k_1 w_1)\end{matrix} \\ w_4(k_1 w_3^2-\mu) & w_2(\mu - k_1 w_3^2) & w_3(3\mu - k_1 w_3^2) & \begin{matrix}k_2 w_4^{-1}\mu(w_3^2+\mu) \\ +w_1(k_1 w_3^2-\mu)\end{matrix} \\ \begin{matrix}-w_3[(k_2+1)\mu \\ +k_1 w_1 w_4]\end{matrix} & \begin{matrix}w_2 w_3(k_2 w_4^{-1}\mu \\ +k_1 w_1)\end{matrix} & \begin{matrix}k_2 w_4^{-1}\mu(w_3^2+\mu) \\ +w_1(k_1 w_3^2-\mu)\end{matrix} & \begin{matrix}-w_3[w_1(2k_2 w_4^{-1}\mu \\ +k_1 w_1) - k_2(k_2-1)w_4^{-2}\mu^2]\end{matrix} \end{bmatrix}. \tag{2.17b}$$

Homogeneity property (2.15) of $f^x(v)$ and $f^y(v)$ implies

$$f_v^x v = sf^x(v); \qquad f_v^y v = sf^y(v). \tag{2.18}$$

For $\alpha = 1 - 2\gamma$, we have $s = 1$ in (2.15b) and (2.18) implies that $f_v^x v = f^x(v), f_v^y v = f^y(v)$. This property may be used in constructing upwind differencing schemes (see [5, 9]). We remark that $\alpha = 1 - 2\gamma < 0$ and therefore v_u is not positive definite; however, the mapping $u \to v$ is one-to-one.

We note that for $\alpha = 1 - 2\gamma$, we have $k_1 = 0$ in (2.16), which results in a great simplification in (2.17):

$f_v^x = -\rho w_4^{-3} \cdot \bar{B}^x$ where

$$\bar{B}^x = \begin{bmatrix} 0 & -w_4^2 & 0 & -(k_2+1)w_2 w_4 \\ -w_4^2 & 3w_2 w_4 & w_3 w_4 & k_2(w_2^2+\mu) - w_1 w_4 \\ 0 & w_3 w_4 & w_2 w_4 & k_2 w_2 w_3 \\ -(k_2+1)\,w_2 w_4 & k_2(w_2^2+\mu) - w_1 w_4 & k_2 w_2 w_3 & -k_2 w_2[2w_1 - (k_2-1)\mu/w_4] \end{bmatrix} \tag{2.19a}$$

$f_v^y = -\rho w_4^{-3} \cdot \bar{B}^y$ where

$$\bar{B}^y = \begin{bmatrix} 0 & 0 & -w_4^2 & -(k_2+1)\,w_3 w_4 \\ 0 & w_3 w_4 & w_2 w_4 & k_2 w_2 w_3 \\ -w_4^2 & w_2 w_4 & 3w_3 w_4 & k_2(w_3^2+\mu) - w_1 w_4 \\ -(k_2+1)\,w_3 w_4 & k_2 w_2 w_3 & k_2(w_3^2+\mu) - w_1 w_4 & -k_2 w_3[2w_1 - (k_2-1)\,\mu/w_4] \end{bmatrix} \tag{2.19b}$$

Here $k_2 = -1 - \gamma/(\gamma - 1)$.

For $\alpha = \gamma > 0$, we have $k_2 = 0$ in (2.16); thus (2.17) becomes

$f_v^x = -\rho\mu^{-1} w_4^{-2} \cdot \hat{B}^x$ where

$$\hat{B}^x = \begin{bmatrix} -k_1 w_2 w_4^2 & w_4(k_1 w_2^2 - \mu) & k_1 w_2 w_3 w_4 & -w_2(\mu + k_1 w_1 w_4) \\ w_4(k_1 w_2^2 - \mu) & -w_2(k_1 w_2^2 - 3\mu) & -w_3(k_1 w_2^2 - \mu) & w_1(k_1 w_2^2 - \mu) \\ k_1 w_2 w_3 w_4 & -w_3(k_1 w_2^2 - \mu) & -w_2(k_1 w_3^2 - \mu) & k_1 w_1 w_2 w_3 \\ -w_2(\mu + k_1 w_1 w_4) & w_1(k_1 w_2^2 - \mu) & k_1 w_1 w_2 w_3 & -k_1 w_1^2 w_2 \end{bmatrix} \tag{2.20a}$$

$f_v^y = -\rho\mu^{-1} w_4^{-2} \cdot \hat{B}^y$ where

$$\hat{B}^y = \begin{bmatrix} -k_1 w_3 w_4^2 & k_1 w_2 w_3 w_4 & w_4(k_1 w_3^2 - \mu) & -w_3(\mu + k_1 w_1 w_4) \\ k_1 w_2 w_3 w_4 & w_3(\mu - k_1 w_2^2) & -w_2(k_1 w_3^2 - \mu) & k_1 w_1 w_2 w_3 \\ w_4(k_1 w_3^2 - \mu) & -w_2(k_1 w_3^2 - \mu) & -w_3(k_1 w_3^2 - 3\mu) & w_1(k_1 w_3^2 - \mu) \\ -w_3(\mu + k_1 w_1 w_4) & k_1 w_1 w_2 w_3 & w_1(k_1 w_3^2 - \mu) & -k_1 w_1^2 w_3 \end{bmatrix} \tag{2.20b}$$

Here $k_1 = 1/\gamma - 2$.

3. Viscosity Terms

In this section, we consider the viscosity terms in the compressible Navier–Stokes equations

$$u_t + [f^x(u)]_x + [f^y(u)]_y = \frac{\partial}{\partial x} Q^x(u, u_x, u_y) + \frac{\partial}{\partial y} Q^y(u, u_x, u_y), \tag{3.1}$$

where $u, f^x(u)$, and $f^y(u)$ are the same as in Section 2, and

$$[Q^x]^T = \{0, \lambda(q_{1x} + q_{2y}) + 2\mu q_{1x}, \mu(q_{2x} + q_{1y}), \mu q_2(q_{1y} + q_{2x}) + \lambda q_1(q_{1x} + q_{2y}) + 2\mu q_1 q_{1x}\}; \tag{3.2a}$$

$$[Q^y]^T = \{0, \mu(q_{1y} + q_{2x}), \lambda(q_{1x} + q_{2y}) + 2\mu q_{2y}, \mu q_1(q_{2x} + q_{1y}) + \lambda q_2(q_{1x} + q_{2y}) + 2\mu q_2 q_{2y}\}; \tag{3.2b}$$

as before, $q_1 = u_2/u_1$ and $q_2 = u_3/u_1$ are the velocity components in the x and y directions, respectively.

Expressing q_1 and q_2 as a function of v in (2.11), we get

$$q_1 = -v_2/v_4, \qquad q_2 = -v_3/v_4,$$

and

$$q_{1x} = v_4^{-2}(-v_4 v_{2x} + v_2 v_{4x}), \qquad q_{2x} = v_4^{-2}(-v_4 v_{3x} + v_3 v_{4x}), \tag{3.3a}$$

$$q_{1y} = v_4^{-2}(-v_4 v_{2y} + v_2 v_{4y}), \qquad q_{2y} = v_4^{-2}(-v_4 v_{3y} + v_3 v_{4y}). \tag{3.3b}$$

Substituting q_{ix}, q_{iy}, $i = 1, 2$ in (3.2) by (3.3), we rewrite (3.2) as

$$Q^x = R^{xx}(v)\, v_x + R^{xy}(v)\, v_y, \tag{3.4a}$$

$$Q^y = R^{yy}(v)\, v_y + R^{yx}(v)\, v_x, \tag{3.4b}$$

where

$$R^{xx}(v) = v_4^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -(\lambda + 2\mu)\, v_4^2 & 0 & (\lambda + 2\mu)\, v_2 v_4 \\ 0 & 0 & -\mu v_4^2 & \mu v_3 v_4 \\ 0 & (\lambda + 2\mu)\, v_2 v_4 & \mu v_3 v_4 & -(\lambda + 2\mu)\, v_2^2 - \mu v_3^2 \end{bmatrix}, \tag{3.5a}$$

$$R^{yy}(v) = v_4^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mu v_4^2 & 0 & \mu v_2 v_4 \\ 0 & 0 & -(\lambda + 2\mu) v_4^2 & (\lambda + 2\mu)\, v_3 v_4 \\ 0 & \mu v_2 v_4 & (\lambda + 2\mu)\, v_3 v_4 & -(\lambda + 2\mu)\, v_3^2 - \mu v_2^2 \end{bmatrix}, \tag{3.5b}$$

$$R^{xy}(v) = v_4^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda v_4^2 & \lambda v_3 v_4 \\ 0 & -\mu v_4^2 & 0 & \mu v_2 v_4 \\ 0 & \mu v_3 v_4 & \lambda v_2 v_4 & -(\lambda + \mu)\, v_3 v_4 \end{bmatrix}, \tag{3.6a}$$

$$R^{yx}(v) = v_4^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu v_4^2 & \mu v_3 v_4 \\ 0 & -\lambda v_4^2 & 0 & \lambda v_2 v_4 \\ 0 & \lambda v_3 v_4 & \mu v_2 v_4 & -(\lambda + \mu)\, v_3 v_4 \end{bmatrix}. \tag{3.6b}$$

We observe that R^{xx} and R^{yy} are symmetric nonnegative matrices (note that $v_4 < 0$ by definition). Matrices R^{xy} and R^{yx} are not symmetric, except in the nonphysical case $\lambda = \mu$; however, $R^{xy} + R^{yx}$ is symmetric, in agreement with [1].

4. Summary

In this paper we have described a symmetric form of systems of conservation laws with entropy. This symmetric form retains the conservation properties of the equations; consequently, weak solutions remain unchanged. In Section 2 we have presented specific symmetric forms of the Euler equations of gas dynamics. Some of these forms are surprisingly simple and therefore are computationally attractive.

We feel that this extra richness of structure is an important feature to be utilized in both the analysis and the numerical approximation of weak solutions of systems of conservation laws with entropy.

Of particular computational interest is the possibility of using the symmetric form

(1.6a) to calculate steady state solutions of (1.1). Here, the term $u_v v_t$ vanishes, and the numerical approximation becomes an iteration method for solutions of the spatial part of (1.6a). The symmetry of the matrix coefficients may provide a way to achieve faster convergence to steady state solutions in this case.

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